Finitistic and Representation Dimensions

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Abstract

Let A be an artin algebra with representation dimension not more than 3. Assume that $_AV$ is an Auslander generator and $M \in \mathrm{add}_A V$ (for example, $M \in \mathrm{add}_A(A \oplus \mathbf{D}A)$), we indicate that both $\mathrm{findim}(\mathrm{End}_A M)$ and $\mathrm{findim}(\mathrm{End}_A M)^{op}$ are finite and consequently, the Gorenstein Symmetry Conjecture, the Wakamatsu-tilting conjecture and the generalized Nakayama conjecture hold for $\mathrm{End}_A M$. In particular, we see that the finitistic dimension conjecture and all the above conjectures hold for artin algebras which can be realized as endomorphism algebras of modules over representation-finite algebras. It is also shown that if every quasi-hereditary algebra has a left idealized extension which is a monomial algebra or an algebra whose representation dimension is not more than 3, then the finitistic dimension conjecture holds.

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Key words: finitistic dimension; representation dimension; representation-finite algebra

1 Introduction

Throughout the paper, we work on artin algebras and finitely generated left modules.

Let A be an artin algebra. Recall that the little finitistic dimension of A, denoted by $\mathrm{findim}A$, is defined to be the supremum of the projective dimensions of all finitely generated modules of finite projective dimension. Similarly, the big finitistic dimension $\mathrm{Findim}A$ is defined, allowing arbitrary A-modules.

In 1960, Bass [4] formulated two so-called finitistic dimension conjectures. The first one asserts that $\mathrm{findim}A = \mathrm{Findim}A$, while the second one claims that $\mathrm{findim}A < \infty$.

It is known that the first finitistic dimension conjecture fails in general [19] and the differences can even be arbitrarily big [14]. However, the second finitistic dimension conjecture is still open. This conjecture is also related to many other homological conjectures (e.g., the Gorenstein Symmetry Conjecture, the Wakamatsu-tilting conjecture and the generalized Nakayama conjecture) and attracts many algebraists, see for instance [1, 8, 16, 20 etc.].

Only few classes of algebras was known to have finite finitistic dimension, see for instance [6, 7, 8, etc.]. In 1991, Auslander and Reiten [3] proved that $\mathrm{findim}A < \infty$ if $\mathcal{P}^{<\infty}$, the category of all A-modules of finite projective dimension, is contravariantly finite. However, Igusa-Smalϕ-Todorov [9] provided an example of artin algebras over which $\mathcal{P}^{<\infty}$ is not contravariantly finite.

In 2002, Angeleri-Hügel and Trlifaj [1] obtained a sufficient and necessary condition for the finiteness of $\mathrm{findim}A$, using the theory of infinitely generated tilting modules. They proved that $\mathrm{findim}A$ is finite if and only if there exists a special tilting A-module (maybe infinitely generated).

In the same year, Igusa and Todorov [10] presented a good way to prove the (second) finitistic dimension conjecture. In particular, they proved that $\mathrm{findim}A$ is finite provided that the representation dimension $\mathrm{repdim}A$, another important homological invariant introduced by Auslander [2], is not more than 3. Recall that $\mathrm{repdim}A = \inf\{\mathrm{gd}(\mathrm{End}_A V) \mid V$ is a generator-cogenerator$\}$, where gd denotes the global dimension and $\mathrm{End}_A V$ denotes the endomorphism algebra of $_A V$. It should be noted that, though the representation dimension of an artin algebra is always finite [11], there is no upper bound for the representation dimension of an artin algebra [13].

Using the Igusa-Todorov functor defined in [10], Xi [15, 16, 17] developed some new ideas to prove the finiteness of finitistic dimension of some artin algebras. For example, it was shown in [16] that, if A is a subalgebra of R such that $\mathrm{rad}A$ is a left ideal in R (in the case R is called a left idealized extension of A), then $\mathrm{findim}A$ is finite provided that $\mathrm{repdim}R$ is not more than 3. In addition, it was proved that the finitistic dimension conjecture holds for all artin algebras if and only if the finiteness of $\mathrm{findim}R$ implies the finiteness of $\mathrm{findim}A$, for any pair A, R such that R is a left idealized extension of A [15].

In this note, we will continue the study of the finitistic dimension conjecture following the ideas comes from [10, 15, 16, 17]. Recall that an Auslander generator over the artin algebra A is a generator-cogenerator $_A V$ such that $\mathrm{repdim}A = \mathrm{gd}(\mathrm{End}_A V)$. Note that it is not known if $\mathrm{findim}A^{op}$ is finite provided that $\mathrm{findim}A$ is finite in general, where A^{op} denotes the opposite algebra of A.

Our main results state as follows.

Theorem 1.1 *Let A be an artin algebra with* $\mathrm{repdim}A \leq 3$. *Assume that $_A V$ is an Aus-*

lander generator. Then both $\mathrm{findim}(\mathrm{End}_A M)$ and $\mathrm{findim}(\mathrm{End}_A M)^{op}$ are finite, whenever $M \in \mathrm{add}_A V$.

Let E be an artin algebra. Recall the following well known conjectures (see for instance [8, 12], etc.).

Gorenstein Symmetry Conjecture $\mathrm{id}(_E E) < \infty$ if and only if $\mathrm{id}(E_E) < \infty$, where id denotes the injective dimension.

Wakamatsu-tilting Conjecture Let $_E\omega$ be a Wakamatsu-tilting module. (1) If $\mathrm{pd}_E\omega < \infty$, then ω is tilting. (2) If $\mathrm{id}_E\omega < \infty$, then ω is cotilting.

Generalized Nakayama Conjecture Each indecomposable injective E-module occurs as a direct summand in the minimal injective resolution of $_E E$.

Note that the Gorenstein symmetry conjecture and the Generalized Nakayama conjecture are special cases of the second Wakamatsu-tilting conjecture. Moreover, if the finitistic dimension conjecture for E and E^{op}, then all the above conjectures hold.

Corollary 1.2 *Let A be an artin algebra with* $\mathrm{repdim}A \leq 3$. *Assume that $_A V$ is an Auslander generator. Then the Gorenstein Symmetry Conjecture, the Wakamatsu-tilting conjecture and the generalized Nakayama conjecture hold for* $\mathrm{End}_A M$.

As a special case of and Theorem 1.1 and Corollary 1.2, we obtain the following result.

Corollary 1.3 *If $A = \mathrm{End}_\Lambda M$ for some module M over a representation-finite algebra Λ, then both $\mathrm{findim}A$ and $\mathrm{findim}A^{op}$ are finite. In particular, the Gorenstein Symmetry Conjecture, the Wakamatsu-tilting conjecture and the generalized Nakayama conjecture hold for A.*

We do not know whether or not every artin algebra can be realized as an endomorphism algebra of some module over a representation-finite algebra. If it is the case, then the above result indicates that the finitistic dimension conjecture holds for all artin algebras.

However, it is well known that every artin algebra can be realized as an endomorphism algebra of a projective and injective module over a quasi-hereditary algebra. Thus the following result shows that, in particular, if every quasi-hereditary algebra has a left idealized extension which is a monomial algebra or an algebra whose representation dimension is not more than 3, then the finitistic dimension conjecture holds.

Proposition 1.4 *Let R be a left idealized extension of A. If $\mathrm{repdim}R \leq 3$ or $\Omega_R^2(R-\mathrm{mod})$ is of finite type (for example, $\mathrm{gd}R \leq 2$), then $\mathrm{findim}(\mathrm{End}_A M)$ is finite, for any projective A-module M.*

2 Finitistic dimension of endomorphism algebras

Let A be an artin algebra. We denoted by $A-$mod the category of all A-modules. If M is an A-module, we use $\mathrm{pd}_A M$ to denote the projective dimension of $_A M$ and use $\Omega_A^i M$ to denote the i-th syzygy of M. Throughout the note, $\mathbf{D}$ denotes the usual duality functor between $A-$mod and $A^{op}-$mod.

The following lemma was well-known, see for instance [17].

Lemma 2.1 *Let A be an artin algebra and let V be a generator-cogenerator in $A-$mod. The following are equivalent for a nonnegative integer n.*

(1) $\mathrm{gd}(\mathrm{End}_A V) \leq n + 2$.

(2) For any $X \in A-$mod, there is an exact sequence $0 \to V_n \to \cdots \to V_1 \to V_0 \to X \to 0$ with each $V_i \in \mathrm{add}_A V$, such that the corresponding sequence induced by the functor $\mathrm{Hom}_A(V, -)$ is also exact.

The following lemma collects some important properties of the Igusa-Todorov functor introduced in [10].

Lemma 2.2 *For any artin algebra A, there is a functor Ψ which is defined on the objects of $A-$mod and takes nonnegative integers as values, such that*

(1) $\Psi(M) = \mathrm{pd}_A M$ provided that $\mathrm{pd}_A M < \infty$.

(2) $\Psi(X) \leq \Psi(Y)$ whenever $\mathrm{add}_A X \subseteq \mathrm{add}_A Y$. The equation holds in case $\mathrm{add}_A X = \mathrm{add}_A Y$.

(3) If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $A-$mod with $\mathrm{pd}_A Z < \infty$, then $\mathrm{pd}_A Z \leq \Psi(X \oplus Y) + 1$.

Let A be an artin algebra and $M \in A-$mod with $E = \mathrm{End}_A M$. Then M is also a right E-module. It is well known that $(M \otimes_E -, \mathrm{Hom}_A(M, -))$ is a pair of adjoint functors and that, for any E-module Y, there is a canonical homomorphism $\sigma_Y : Y \to \mathrm{Hom}_A(M, M \otimes_E Y)$ definied by $n \to [t \to t \otimes n]$. It is easy to see that σ_Y is an isomorphism provided that Y is a projective E-module.

The following lemma is elementary but essential to prove our results.

Lemma 2.3 *Let $M \in A-$mod and $E = \mathrm{End}_A M$. Then, for any $X \in E-$mod, $\Omega_E^2 X \simeq \mathrm{Hom}_A(M, Y)$ for some $Y \in A-$mod.*

Proof. Obviously, we have an exact sequence

$$0 \to \Omega_E^2 X \to E_1 \to E_0 \to X \to 0$$

with $E_0, E_1 \in E-$mod projective. Applying the functor $M \otimes_E -$, we obtain an induced exact sequence

$$0 \to Y \to M \otimes_E E_1 \to M \otimes_E E_0 \to M \otimes_E X \to 0,$$

for some $Y \in A-$mod. Now applying the functor $\mathrm{Hom}_A(M, -)$, we further have an induced exact sequence

$$0 \to \mathrm{Hom}_A(M, Y) \to \mathrm{Hom}_A(M, M \otimes_E E_1) \to \mathrm{Hom}_A(M, M \otimes_E E_0).$$

Moreover, there is the following commutative diagram

$$
\begin{array}{ccccc}
0 \longrightarrow & \Omega_E^2 X & \longrightarrow & E_1 & \longrightarrow & E_0 \\
& \downarrow \phi & & \downarrow \sigma_{E_1} & & \downarrow \sigma_{E_0} \\
0 \longrightarrow & \mathrm{Hom}_A(M, Y) & \longrightarrow & \mathrm{Hom}_A(M, M \otimes_E E_1) & \longrightarrow & \mathrm{Hom}_A(M, M \otimes_E E_0).
\end{array}
$$

Since $E = \mathrm{End}_A M$ and $E_0, E_1 \in \mathrm{add}_E E$, the canonical homomorphisms σ_{E_0} and σ_{E_1} are isomorphisms. It follows that $\Omega_E^2 X \simeq \mathrm{Hom}_A(M, Y)$.

$\#$

Theorem 2.4 *Let A be an artin algebra and $_A V$ be a generator-cogenerator in $A-$mod such that $\mathrm{gd}(\mathrm{End}_A V) \leq 3$. Then $\mathrm{findim} E$ is finite, where $E = \mathrm{End}_A M$ for some $M \in \mathrm{add}_A V$.*

Proof. Suppose that $X \in E-$mod and that $\mathrm{pd}_E X < \infty$. Then $\mathrm{pd}_E(\Omega_E^2 X) < \infty$. Moreover, $\Omega_E^2 X \simeq \mathrm{Hom}_A(M, Y)$ for some $Y \in A-$mod, by Lemma 2.3. Since $_A V$ is a generator and $\mathrm{gd}(\mathrm{End}_A V) \leq 3$, by Lemma 2.1 we obtain an exact sequence

$$0 \to V_1 \to V_0 \to Y \to 0 \ (\dagger)$$

with $V_0, V_1 \in \mathrm{add}_A V$ such that the corresponding sequence induced by the functor $\mathrm{Hom}_A(V, -)$ is also exact. Note that $M \in \mathrm{add}_A V$, so the sequence $(\dagger)$ also stays exact under the functor $\mathrm{Hom}_A(M, -)$. Thus, we have the following exact sequence in $E-$mod:

$$0 \to \mathrm{Hom}_A(M, V_1) \to \mathrm{Hom}_A(M, V_0) \to \mathrm{Hom}_A(M, Y) \to 0.$$

Now by Lemma 2.2, we deduce that

$$
\begin{aligned}
\mathrm{pd}_E X &\leq \mathrm{pd}_E(\Omega_E^2 X) + 2 \\
&= \mathrm{pd}_E(\mathrm{Hom}_A(M, Y)) + 2 \\
&\leq \Psi(\mathrm{Hom}_A(M, V_0) \oplus \mathrm{Hom}_A(M, V_1)) + 1 + 2 \\
&\leq \Psi(\mathrm{Hom}_A(M, V)) + 1 + 2 < \infty.
\end{aligned}
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It is not known if findimA^{op} is finite provided that findim$A$ is finite in general. For an artin algebra A with repdim$A \leq 3$, it is known that findim$A$ is finite [11] and since repdim$A = $ repdimA^{op}, findimA^{op} is also finite in the case. The following corollary is also a generalization of this fact.

Corollary 2.5 *Let A be an artin algebra with* repdim$A \leq 3$. *Assume that ${}_A V$ is an Auslander generator. If $M \in \text{add}_A V$, then both* findim$(\text{End}_A M)$ *and* findim$(\text{End}_A M)^{op}$ *are finite. In particular, the result holds for any ${}_A M \in \text{add}_A(A \oplus \mathbf{D}A)$.*

Proof. If ${}_A V$ is an Auslander generator in $A-$mod, then ${}_{A^{op}} \mathbf{D}V$ is an Auslander generator in $A^{op}-$mod. Moreover, if $M \in \text{add}_A V$, then $\mathbf{D}M \in \text{add}_{A^{op}} \mathbf{D}V$. Obviously, $(\text{End}_A M)^{op} \simeq \text{End}_{A^{op}} \mathbf{D}M$. Now the conclusion follows from Theorem 2.4.

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Since all representation-finite algebras have representation dimension 2, we obtain the following corollary.

Corollary 2.6 *If $A = \text{End}_\Lambda M$ for some representation-finite algebra Λ and some $M \in \Lambda-$mod, then both* findim$A$ *and* findimA^{op} *are finite.*

Proof. Since Λ is of finite representation type, the module $V = \oplus_{i=1}^n V_i$ is an Auslander generator, where $\{V_i, i = 1, \cdots, n\}$ is the representation set of all indecomposable Λ-modules. Now the conclusion follows from Corollary 2.5.

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The above result suggests a strong connection between the finitistic dimension conjecture and the following endomorphism algebra realization problem of artin algebras.

Problem 1 Can all artin algebras be realized as endomorphism algebras of modules over representation-finite algebras? If not, what algebra has such a realization?

Obviously, the affirmative answer to the first question implies the finitistic dimension conjecture holds, by Corollary 2.6.

In contrast, it is well known that every artin algebra can be realized as an endomorphism algebra of a projective and injective module over a quasi-hereditary algebra.

Let A, R be both Artin algebras. Following [15], we say that R is a left idealized extension of A if $A \subseteq R$ has the same identity and rad$A$ is a left ideal in R.

Proposition 2.7 *Let R be a left idealized extension of A. If* repdim$R \leq 3$ *or $\Omega_R^2(R-$mod$)$ is of finite type (for example,* gd$R \leq 2$*), then* findim$E$ *is finite, where $E = \text{End}_A M$ for some projective A-module M.*

Proof. Suppose that $X \in E-\text{mod}$ and that $\text{pd}_E X < \infty$. Then $\text{pd}_E(\Omega^2_E X) < \infty$. Moreover, $\Omega^2_E X \simeq \text{Hom}_A(M, Y)$ for some $Y \in A-\text{mod}$, by Lemma 2.3. Since $_A M$ is projective, the proof of Lemma 2.3 in fact indicates that $Y \in \Omega^2_A(A-\text{mod})$. Then $Y \simeq \Omega_R Z \oplus Q$ as R-modules for some $Z, Q \in R-\text{mod}$ with Q projective, by [15, Erratum]. If $\text{repdim} R \leq 3$ or $\Omega^2_R(R-\text{mod})$ is of finite type, we can obtain an exact sequence of R-modules

$$0 \to V_1 \to V_0 \to Y \to 0 \ (\ddagger)$$

with $V_0, V_1 \in \text{add}_R V$. Here $_R V$ is an Auslander generator in case $\text{repdim} R \leq 3$ or $V = R \oplus N$ in case $\Omega^2_R(R-\text{mod})$ is of finite type, where N is the direct sum of nonisomorphic indecomposable R-modules appeared in $\Omega^2_R(R-\text{mod})$. The exact sequence $(\ddagger)$ then restricts to an exact sequence in $A-\text{mod}$. Since $_A M$ is projective, we have the following exact sequence in $E-\text{mod}$:

$$0 \to \text{Hom}_A(M, V_1) \to \text{Hom}_A(M, V_0) \to \text{Hom}_A(M, Y) \to 0.$$

Now by Lemma 2.2, we deduce that

$$\begin{aligned}
\text{pd}_E X &\leq \text{pd}_E(\Omega^2_E X) + 2 \\
&= \text{pd}_E(\text{Hom}_A(M, Y)) + 2 \\
&\leq \Psi(\text{Hom}_A(M, V_0) \oplus \text{Hom}_A(M, V_1)) + 1 + 2 \\
&\leq \Psi(\text{Hom}_A(M, V)) + 1 + 2 < \infty.
\end{aligned}$$

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The above result has the following corollary which contains [18, Corollary 2.15], where the result was proved under an additional condition.

Corollary 2.8 *If A has a left idealized extension which is a monomial algebra, then $\text{findim} E$ is finite, where $E = \text{End}_A M$ for some projective A-module M.*

Proof. It was noted that $\Omega^2_R(R-\text{mod})$ is of finite type in case R is a finite dimensional monomial relation algebra, see [5, Example 2.3(a)] (also [19, Theorem A]). Thus the conclusion follows from Proposition 2.7.

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Problem 2 Does every quasi-hereditary algebra have a left idealized extension which is a monomial algebra or an algebra whose representation dimension is not more than 3?

Of course, the affirmative answer to the problem also implies the finitistic dimension conjecture holds.

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